ON THE INTERSECTION OF ANNIHILATOR OF THE VALABREGA-VALLA MODULE

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ABSTRACT. Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring with an infinite residue field and let I be an $\mathfrak{m}$-primary ideal. Let $\mathbf{x} = x_1,\ldots,x_r$ be a A-superficial sequence with respect to I. Set

$$\mathcal{V}_I(\mathbf{x}) = \bigoplus_{n\geq 1} \frac{I^{n+1}\cap(\mathbf{x})}{\mathbf{x}I^n}.$$

A consequence of a theorem due to Valabrega and Valla is that $\mathcal{V}_I(\mathbf{x}) = 0$ if and only if the initial forms $x_1^*,\ldots,x_r^*$ is a $G_I(A)$ regular sequence. Furthermore this holds if and only if depth $G_I(A) \geq r$. We show that if depth $G_I(A) < r$ then

$$\mathfrak{a}_r(I) = \bigcap_{\substack{\mathbf{x} = x_1,\ldots,x_r \text{ is a} \\ A\text{-superficial sequence w.r.t } I}} \operatorname{ann}_A \mathcal{V}_I(\mathbf{x}) \quad \text{is } \mathfrak{m}\text{-primary.}$$

Suprisingly we also prove that under the same hypotheses,

$$\bigcap_{n\geq 1} \mathfrak{a}_r(I^n) \quad \text{is also } \mathfrak{m}\text{-primary.}$$

INTRODUCTION

Let $(A,\mathfrak{m})$ be a Noetherian local ring with an infinite residue field. The notion of minimal reduction of an ideal I in A was discovered more than fifty years ago by Northcott and Rees; [10]. It plays an essential role in the study of blow-up algebra's. Nevertheless minimal reductions are highly non-unique. The intersection of all minimal reductions is named as *core* of I and denoted by core(I). This was introduced by Rees and Sally in [11]. It has been extensively investigated in

[4],[5] and [9]. When A is Cohen-Macaulay and I is $\mathfrak{m}$-primary; Rees and Sally proved that core(I) is again $\mathfrak{m}$-primary and so is a finite intersection. In this paper we study a different intersection of ideals.

Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with an infinite residue field and let I be an $\mathfrak{m}$-primary ideal. Let $\mathbf{x} = x_1,\ldots,x_r$ be a A-superficial sequence with respect to I. Set

$$\mathcal{V}_I(\mathbf{x}) = \bigoplus_{n\geq 1} \frac{I^{n+1}\cap(\mathbf{x})}{\mathbf{x}I^n}.$$

We call $\mathcal{V}_I(\mathbf{x})$ the Valabrega-Valla module of I with respect to $\mathbf{x}$. A consequence of a theorem due to Valabrega and Valla, [13, 2.3] is that $\mathcal{V}_I(\mathbf{x}) = 0$ if and only if

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the initial forms $x_1^*, \ldots, x_r^*$ is a $G_I(A)$ regular sequence. Furthermore this holds if and only if $\operatorname{depth} G_I(A) \geq r$, see [7, 2.1]. In general notice each $\mathcal{V}_I(\mathbf{x})$ has finite length and so $\operatorname{ann}_A \mathcal{V}_I(\mathbf{x})$ is $\mathfrak{m}$-primary. We prove, see Theorem 5.3, that

$$\mathfrak{a}_r(I) = \bigcap_{\substack{\mathbf{x} = x_1, \ldots, x_r \text{ is a} \\ A\text{-superficial sequence w.r.t } I}} \operatorname{ann}_A \mathcal{V}_I(\mathbf{x}) \quad \text{is } \mathfrak{m}\text{-primary.}$$

Our intersection of ideals is in some sense analogous to that of core of I; since notice that

$$\operatorname{core}(I) = \bigcap_{\substack{J \text{ minimal} \\ \text{reduction of } I}} \operatorname{ann}_A \frac{A}{J}.$$

Nevertheless they are two different invariants of I. Furthermore our techniques are totally different from that in the papers listed above.

By a result of Elias $\operatorname{depth} G_{I^n}(A)$ is constant for all $n \gg 0$, see [6, 2.2]. Since $\operatorname{core}(I) \subseteq I$ we have $\bigcap_{n\geq 1} \operatorname{core}(I^n) = 0$. Suprisingly, see Theorem 6.3, we have that if $\operatorname{depth} G_I(A) < r$ then

$$\bigcap_{n\geq 1} \mathfrak{a}_r(I^n) \quad \text{is } \mathfrak{m}\text{-primary.}$$

We now assume A is also complete. Let $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n$ be the Rees algebra of I. Set $L = L^I(A) = \bigoplus_{n\geq 0} A/I^{n+1}$. It can be shown easily that L is a $\mathcal{R}(I)$-module. Of course L is not finitely generated as a $\mathcal{R}(I)$-module. Nevertheless we prove that its local cohomology modules $H^i_{\mathcal{R}(I)_+}(L)$ are *-Artinian for $i = 0, \ldots, d-1$; see Theorem 4.3. Recall a graded $\mathcal{R}(I)$-module N is said to be $*$-Artininan if it satisfies d.c.c on its graded submodules. Set $\mathfrak{b}_i(I) = \operatorname{ann}_{\mathcal{R}(I)} H^i_{\mathcal{R}(I)_+}(L)$ for $i = 0, \ldots, d-1$ and set $\mathfrak{q}_i(I) = \mathfrak{b}_i(I) \cap A$. Since $H^i_{\mathcal{R}(I)_+}(L)$ is *-Artinian; it is not so difficult to show that $\mathfrak{q}_i$ is $\mathfrak{m}$-primary (or equal to A); see Corollary 4.4.

In Theorem 5.2 we prove that

$$\mathfrak{a}_r(I) \supseteq \mathfrak{q}_0(I)\mathfrak{q}_1(I)\cdots\mathfrak{q}_{r-1}(I).$$

Next note that $L^I(A)(-1)$ behaves well with respect to the Veronese functor. Clearly

$$\left(L^I(A)(-1)\right)^{<l>} = L^{I^l}(A)(-1) \quad \text{for each } l \geq 1.$$

Also local cohomolgy commutes with the Veronese functor. As a consequence we have

$$\mathfrak{q}_i(I^l) \supseteq \mathfrak{q}_i(I) \quad \text{for each } l \geq 1 \text{ and } i = 0, 1, \ldots, r-1.$$

It follows that

$$\bigcap_{n\geq 1} \mathfrak{a}_r(I^n) \supseteq \mathfrak{q}_0(I)\mathfrak{q}_1(I)\cdots\mathfrak{q}_{r-1}(I).$$

The $\mathcal{R}(I)$-module $L^I(A)$ is not finitely generated $\mathcal{R}(I)$-module. However it is quasi-finite $\mathcal{R}(I)$-module, see section 1.5. Quasi-finite module were introduced in [8, page 10]. Surprisingly we were able to prove that if E is a quasi-finite $\mathcal{R}(I)$-module and has a filter-regular sequence of length s then the local cohomology modules $H^i_{\mathcal{R}(I)_+}(E)$ are all *-Artinian for $i = 0, \ldots, s-1$.

We also study the Koszul homology of a quasi-finite module with respect to a filter regular sequence. We then use a spectral sequence, first used by P. Roberts [12, Theorem 1], to relate cohomological annihilators with that of annihilators of

the Koszul complex. We however have to very careful in our proof since we are dealing with infinitely generated modules.

We now describe in brief the contents of this paper. In section 1 we introduce notation and discuss a few preliminary facts that we need. In section 2 we study a few basic properties of $L^I(M)$. In section 3 we prove some properties of Koszul homology of quasi-finite modules with respect to filter-regular sequence. We also compute $H_1(\mathbf{u}, L^I(M))$ where $\mathbf{u} = x_1t, \ldots, x_rt \in \mathcal{R}(I)_1$ is a $L^I(M)$-filter regular sequence. In section 4 we study local cohomology of quasi-finite modules E with $\ell(E_n)$ finite for all $n \in \mathbb{Z}$. In section 5 we prove that $\mathfrak{a}_r(I)$ is $\mathfrak{m}$-primary (or A). In section 6 we show that $\bigcap_{n\geq 1} \mathfrak{a}_r(I^n)$ is $\mathfrak{m}$-primary (or A).

1. Notation and Preliminaries

Throughout we assume that $(A, \mathfrak{m})$ is a Noetherian local ring with an infinite residue field $k = A/\mathfrak{m}$. Let M be a finitely generated A-module of dimension r and let I be an ideal of definition for M; i.e, $\ell(M/IM)$ is finite. Here $\ell(-)$ denotes length. For undefined terms see [3], especially sections 4.5 and 4.6.

1.1. Assume $r = \dim M \geq 1$. Let $x \in I \setminus I^2$. We say x is M-*superficial* with respect to I if for some $c \geq 1$ we have $(I^{n+1}M \colon x) \cap I^cM = I^nM$ for all $n \gg 0$. If $\operatorname{depth} M > 0$ then using the Artin-Rees Lemma one can prove that $(I^{n+1}M \colon x) = I^nM$ for all $n \gg 0$.

Superficial sequences can be defined as usual. Since k is infinite M-superficial sequences of length $r = \dim M$ exists.

1.2. Let $\mathbf{x} = x_1, \ldots, x_r$ be a M-superficial sequence with respect to I. The Valabrega-Valla module of I with respect to M and $\mathbf{x}$ is

$$\mathcal{V}_I(\mathbf{x}, M) = \bigoplus_{n\geq 1} \frac{I^{n+1}M \cap \mathbf{x}M}{\mathbf{x}I^nM}.$$

We consider it as a A-module. Set $\mathcal{V}_I(\mathbf{x}) = \mathcal{V}_I(\mathbf{x}, A)$.

1.3. Let $\widehat{\mathcal{R}}(I) = \bigoplus_{n\in\mathbb{Z}} I^nt^n$ denote the extended Rees-algebra of A with respect to I. Here $I^n = A$ for $n \leq 0$. We consider it as a subring of $A[t, t^{-1}]$. Let $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^nt^n$ denote the Rees-algebra of A with respect to I. We consider it as a subring of $A[t]$. Of course we can consider $\mathcal{R}(I)$ as a subring of $\widehat{\mathcal{R}}(I)$ too. Both these embedding's of $\mathcal{R}(I)$ would be useful for us. Set

$$\widehat{\mathcal{R}}(I)_M = \bigoplus_{n\in\mathbb{Z}} I^nMt^n \quad \text{and} \quad \mathcal{R}(I)_M = \bigoplus_{n\geq 0} I^nMt^n.$$

We call $\widehat{\mathcal{R}}(I)_M$ the extended Rees module of M with respect to I and we call $\mathcal{R}(I)_M$ to be the Rees module of M with respect to I.

1.4. Consider $L^I(M) = \bigoplus_{n\geq 0} M/I^{n+1}M$. We consider $L^I(M)$ as a $\widehat{\mathcal{R}}(I)$-module as follows:

Consider the exact sequence

$$0 \longrightarrow \widehat{\mathcal{R}}(I)_M \longrightarrow M[t, t^{-1}] \longrightarrow L^I(M)(-1) \longrightarrow 0.$$

Here $M[t, t^{-1}] = M \otimes_A A[t, t^{-1}]$. This exact sequence gives $L^I(M)$ a structure of $\widehat{\mathcal{R}}(I)$-module. Since $\mathcal{R}(I)$ is a subring of $\widehat{\mathcal{R}}(I)$; we also get that $L^I(M)$ is a

$\mathcal{R}(I)$-module. We may also see this directly through the exact sequence

$$0 \longrightarrow \mathcal{R}(I)_M \longrightarrow M[t] \longrightarrow L^I(M)(-1) \longrightarrow 0$$

1.5. Quasi-finite modules It will be convenient at times to work a little more generally. We extend definition of quasi-finite modules from that of [8, page 10]. Let $E = \bigoplus_{n\in\mathbb{Z}} E_n$ be a $\mathcal{R}(I)$-module. We say E is *quasi-finite of order at least* s if

(1) E_n is a finitely generated A-module for all $n \in \mathbb{Z}$
(2) $E_n = 0$ for all $n \ll 0$.
(3) For $i = 0, \ldots, s-1$ we have $H^i_{\mathcal{R}(I)_+}(E)_n = 0$ for all $n \gg 0$.

Remark 1.6. Of course if E is a finitely generated $\mathcal{R}(I)$-module then it is quasi-finite of any order $s \geq 1$. In the next section we prove that if M is Cohen-Macaulay of dimension $r \geq 1$ and I is an ideal of definition for M then $L^I(M)$ is quasi-finite of order at-least r.

1.7. Let $E = \bigoplus_{n\in\mathbb{Z}} E_n$ be a non-necessarily finitely generated $\mathcal{R}(I)$-module with $E_n = 0$ for all $n \ll 0$. An element $u \in \mathcal{R}(I)_1$ is said to be *E-filter regular* if $(0 \colon {}_E u)_n = 0$ for all $n \gg 0$.

Remark 1.8. If E is quasi-finite of order at-least $s(\geq 2)$ and u is E-filter regular then E/uE is quasi-finite of order at-least $s-1$. This can be proved by noting that $(0 \colon {}_E u)$ is $\mathcal{R}(I)_+$-torsion.

1.9. Let $E = \bigoplus_{n\in\mathbb{Z}} E_n$ be a quasi-finite $\mathcal{R}(I)$-module of order at-least s. Let $\mathbf{u} = u_1, \ldots, u_r \in \mathcal{R}(I)_1$ be a sequence and assume $r \leq s$. We say $\mathbf{u}$ is a E-filter regular sequence if u_1 is E-filter regular, u_2 is E/u_1E-filter-regular, …, u_r is $E/(u_1, \ldots, u_{r-1})E$ filter-regular.

Proposition 1.10. *Assume that the residue field of A is uncountable. Let E be a quasi-finite $\mathcal{R}(I)$-module of order at least s. Then there exists $\mathbf{u} = u_1, \ldots, u_s \in \mathcal{R}(I)_1$ which is E-filter regular sequence.*

Proof. It is sufficient to do this for $s = 1$. In this case the result follows from [8, 2.7] □

Remark 1.11. Assume M is Cohen-Macaulay. Let $\mathbf{x} = x_1, \ldots, x_r$ be a M-superficial sequence with respect to I. Set $u_i = x_i t \in \mathcal{R}(I)_1$ for $i = 1, \ldots, r$. In the next section we show that $\mathbf{u} = u_1, \ldots, u_r$ is a $L^I(M)$ filter-regular sequence. We do not need the residue field of A to be uncountable.

2. $L^I(M)$

2.1. Setup and Introduction: In this section M is a *Cohen-Macaulay* A-module of dimension $r \geq 1$ and I is an ideal of definition for M. We consider the $\widehat{\mathcal{R}}(I)$-module $L^I(M) = \bigoplus_{n\geq 0} M/I^{n+1}M$. We prove that $L^I(M)$ is a quasi-finite $\mathcal{R}(I)$-module of order at least r. Let $\mathbf{x} = x_1, \ldots, x_r$ be a M-superficial sequence with respect to I. Set $u_i = x_i t \in \mathcal{R}(I)_1$ for $i = 1, \ldots, r$. We also show that $\mathbf{u} = u_1, \ldots, u_r$ is a $L^I(M)$ filter-regular sequence.

2.2. If E is a graded $\widehat{\mathcal{R}}(I)$-module then notice that

$$H^i_{\mathcal{R}(I)_+}(E) \cong H^i_{\widehat{\mathcal{R}}(I)_+}(E) \quad \text{as } \mathcal{R}(I)\text{-modules.}$$

Note that $\widehat{\mathcal{R}}(I)_+$ denotes the ideal $\mathcal{R}(I)_+\widehat{\mathcal{R}}(I)$ of $\widehat{\mathcal{R}}(I)$. The following result is known when $M = A$; see [1, 3.8].

Lemma 2.3. *[with hypotheses as in 2.1] As $\mathcal{R}(I)$-modules:*

(1) $H^1_{\widehat{\mathcal{R}}(I)_+}(\widehat{\mathcal{R}}(I)_M)$ *is a quotient of* $H^1_{\mathcal{R}(I)_+}(\mathcal{R}(I)_M)$.

(2) $H^i_{\widehat{\mathcal{R}}(I)_+}(\widehat{\mathcal{R}}(I)_M) \cong H^i_{\mathcal{R}(I)_+}(\mathcal{R}(I)_M)$ *for* $i \geq 2$.

Proof. (Sketch) We use 2.2 and the following short exact sequence of $\mathcal{R}(I)$-modules

$$0 \longrightarrow \mathcal{R}(I)_M \longrightarrow \widehat{\mathcal{R}}(I)_M \longrightarrow \widehat{\mathcal{R}}(I)_M/\mathcal{R}(I)_M \longrightarrow 0.$$

Notice $\widehat{\mathcal{R}}(I)_M/\mathcal{R}(I)_M$ is $\mathcal{R}(I)_+$-torsion. □

Proposition 2.4. $L^I(M)$ *is quasi-finite of order* $r = \dim M$.

Proof. Set $L = L^I(M)$. Notice $H^i_{\mathcal{R}(I)_+}(L) = H^i_{\widehat{\mathcal{R}}(I)_+}(L)$ as $\mathcal{R}(I)$-modules. Let $\mathbf{x} = x_1, \dots, x_r$ be a M-superficial sequence with respect to I. Set $u_i = x_i t \in \mathcal{R}(I)_1$ for $i = 1, \dots, r$.

Let $\mathbf{x} = x_1, \dots, x_r$ be a M-superficial sequence with respect to I. Set $u_i = x_i t \in \mathcal{R}(I)_1$ for $i = 1, \dots, r$. It can be easily checked that $\mathbf{u}$ is a $M[t, t^{-1}]$ regular sequence. So $H^i_{\widehat{\mathcal{R}}(I)_+}(M[t, t^{-1}]) = 0$ for $i = 0, \dots, r-1$.

We consider the exact sequence

$$0 \longrightarrow \widehat{\mathcal{R}}(I)_M \longrightarrow M[t, t^{-1}] \longrightarrow L(-1) \longrightarrow 0.$$

Taking local cohomology with respect to $\widehat{\mathcal{R}}(I)_+$ we get that

(a) $H^i_{\widehat{\mathcal{R}}(I)_+}(L(-1)) \cong H^{i+1}_{\widehat{\mathcal{R}}(I)_+}(\widehat{\mathcal{R}}(I)_M)$ for $i = 0, \dots, r-2$.

(b) $H^{r-1}_{\widehat{\mathcal{R}}(I)_+}(L(-1))$ is a submodule of $H^r_{\widehat{\mathcal{R}}(I)_+}(\widehat{\mathcal{R}}(I)_M)$.

The result now follows from Lemma 2.3, Remark 2.2 and [2, 15.1.5]. □

Proposition 2.5. *Let* $\mathbf{x} = x_1, \dots, x_r$ *be a* M*-superficial sequence with respect to* I*. Set* $u_i = x_i t \in \mathcal{R}(I)_1$ *for* $i = 1, \dots, r$*. Then* $\mathbf{u}$ *is a* $L^I(M)$ *filter-regular sequence.*

Proof. Set $L = L^I(M)$. We first show that u_1 is L filter regular. Notice

$$(0 \colon {}_L u_1) = \bigoplus_{n \geq 0} \frac{I^{n+1}M \colon {}_M x_1}{I^n M}.$$

Since x_1 is M-superficial it follows that u_1 is L filter regular; see 1.1.

Check that

$$\frac{L}{u_1 L} = \bigoplus_{n \geq 0} \frac{M}{x_1 M + I^{n+1} M} = L^I(M/x_1 M).$$

The result now follows from an easy induction on $\dim M$. □

3. Koszul homology of quasi-finite modules with respect to filter-regular sequence

In this section we show some properties of Koszul homology of a quasi-finite module with respect to a filter regular sequence. We also compute the Koszul homology of $L^I(M)$ with respect to $\mathbf{u} = x_1 t, \dots, x_s t$ where $x_1, \dots, x_s$ is an M-superficial sequence with respect to I.

Theorem 3.1. *Let* E *be a quasi-finite* $\mathcal{R}(I)$*-module of order at least* s *and let* $\mathbf{u} = u_1, \dots, u_s$ *be a* E*-filter regular sequence. Then for* $i = 1, \dots, s$ *we have*

(1) $H_i(\mathbf{u}, E)$ *is a finitely generated* $\mathcal{R}(I)$*-module. It is also* $\mathcal{R}(I)_+$*-torsion. In particular* $H_i(\mathbf{u}, E)$ *is a finitely generated* A*-module.*

(2) *If* $\mathbf{u}$ *is* E*-regular sequence then* $H_i(\mathbf{u}, E) = 0$ *for* $i = 1, \ldots, s$.
(3) *If* $H_1(\mathbf{u}, E) = 0$ *then* $\mathbf{u}$ *is a* E*-regular sequence.*

Proof. (1) We prove it by induction on s.

The case $s = 1$.
Notice $H_1(u_1, E) = (0 \colon_{u_1} E)$. Since u_1 is E-filter regular we get that $H_1(u_1, E)$ is a finitely generated A-module and hence a finitely generated $\mathcal{R}(I)$-module. Clearly it is also $\mathcal{R}(I)_+$ torsion.

We assume the result for $s = r$ and prove for $s = r + 1$. Let $\mathbf{u} = u_1, \ldots, u_r, u_{r+1}$ and $\mathbf{u}' = u_1, \ldots, u_r$. We have for all $i \geq 0$ an exact sequence

$$(3.1.1) \quad 0 \longrightarrow H_0(u_{r+1}, H_i(\mathbf{u}', E)) \longrightarrow H_i(\mathbf{u}, E) \longrightarrow H_1(u_{r+1}, H_{i-1}(\mathbf{u}', E)) \longrightarrow 0$$

Using induction hypothesis it follows that for $i \geq 2$ the modules $H_i(\mathbf{u}, E)$ are finitely generated $\mathcal{R}(I)$-modules and also $\mathcal{R}(I)_+$-torsion. For $i = 1$ notice that

(a) $H_0(u_{r+1}, H_1(\mathbf{u}', E))$ is finitely generated $\mathcal{R}(I)$-module. It is also $\mathcal{R}(I)_+$-torsion.

(b) $H_1(u_{r+1}, H_0(\mathbf{u}', E)) = H_1(u_{r+1}, E/\mathbf{u}'E)$. Since u_{r+1} is $E/\mathbf{u}'E$-filter regular then by $s = 1$ case we have that $H_1(u_{r+1}, H_0(\mathbf{u}', E))$ is a finitely generated $\mathcal{R}(I)$-module and it also $\mathcal{R}(I)_+$-torsion

The result follows.

(2) The standard proof works.

(3) Nothing to prove when $s = 1$. So assume $s \geq 2$. Set $r = s - 1$. We use equation 3.1.1. If $H_1(\mathbf{u}, E) = 0$ then $H_0(u_{r+1}, H_1(\mathbf{u}', E)) = 0$. So we have $H_1(\mathbf{u}', E) = u_{r+1} H_1(\mathbf{u}', E)$. Since $H_1(\mathbf{u}', E)$ is a finitely generated graded $\mathcal{R}(I)$-module and u_{r+1} has positive degree it follows that $H_1(\mathbf{u}', E) = 0$. By induction hypothesis it follows that $u_1, \ldots, u_r$ is a E-regular sequence.

From 3.1.1 we also get

$$H_1(u_{r+1}, H_0(\mathbf{u}', E)) = H_1(u_{r+1}, E/\mathbf{u}'E) = 0.$$

So u_{r+1} is $E/\mathbf{u}'E$- regular. It follows that $\mathbf{u}$ is a E-regular sequence. □

Proposition 3.2. *Let* M *be a Cohen-Macaulay* A*-module of dimension* $r \geq 1$ *and let* I *be an ideal of definition for* M. *Let* $\mathbf{x} = x_1, \ldots, x_s$ *be a* M*-superficial sequence with respect to* I *with* $s \leq r$. *Set* $u_i = x_i t \in \mathcal{R}(I)_1$ *for* $i = 1, \ldots, s$. *Then* $\mathbf{u}$ *is a* $L^I(M)$ *filter-regular sequence and*

$$H_1(\mathbf{u}, L^I(M)) = \bigoplus_{n \geq 1} \frac{I^{n+1}M \cap \mathbf{x}M}{\mathbf{x}I^n M} = \mathcal{V}_I(\mathbf{x}, M).$$

Proof. Set $L = L^I(M)$. In 2.5 we have shown already that $\mathbf{u}$ is a $L^I(M)$ filter-regular sequence.

Consider the exact sequence

$$0 \longrightarrow \widehat{\mathcal{R}}(I)_M \longrightarrow M[t, t^{-1}] \longrightarrow L(-1) \longrightarrow 0.$$

It can be easily checked that $\mathbf{u}$ is a $M[t, t^{-1}]$ regular sequence. So $H_1(\mathbf{u}, M[t, t^{-1}]) = 0$. Thus we have an exact sequence

$$0 \longrightarrow H_1(\mathbf{u}, L(-1)) \longrightarrow H_0(\mathbf{u}, \widehat{\mathcal{R}}(I)_M) \longrightarrow H_0(\mathbf{u}, M[t, t^{-1}]) \longrightarrow H_0(\mathbf{u}, L) \longrightarrow 0.$$

Notice

$$H_0(\mathbf{u}, \widehat{\mathcal{R}}(I)_M) = \bigoplus_{n \in Z} \frac{I^n M}{\mathbf{x}I^{n-1}M} \quad \text{and} \quad H_0(\mathbf{u}, M[t, t^{-1}]) = M/\mathbf{x}M[t, t^{-1}]$$

So

$$H_1(\mathbf{u}, L(-1)) = \bigoplus_{n \in Z} \frac{I^n M \cap \mathbf{x}M}{\mathbf{x}I^{n-1}M}.$$

The result follows. $\square$

4. LOCAL COHOMOLOGY OF QUASI-FINITE MODULES E WITH $\ell(E_n)$ FINITE FOR ALL $n \in \mathbb{Z}$

In this section we prove a suprising fact: the local cohomology modules $H^i_{\mathcal{R}(I)_+}(L^I(M))$ are all *-Artinian for $i = 0, \ldots, \operatorname{depth} M - 1$. It is convenient to prove it in the generality of quasi-finite modules.

4.1. Throughout this section $H^i(-) = H^i_{\mathcal{R}(I)_+}(-)$ the i-th local cohomology functor with respect to $\mathcal{R}(I)_+$. In this section we assume that

1. $(A, \mathfrak{m})$ is complete with infinite residue field.
2. E is a quasi-finite module of order at least s.
3. There exists an E-filter regular sequence of length s.
4. $\ell(E_n)$ finite for all $n \in \mathbb{Z}$.

Remark 4.2. The hypothesis on existence of E-filter regular sequence of length s is automatically satisfied if k is uncountable. The assumption "$\ell(E_n)$ finite for all $n \in \mathbb{Z}$" is to imitate that of $L^I(M)$. Finally if M is CM and A has infinite residue field then assumptions 2, 3, 4 are automatically satisfied for $L^I(M)$. The assumption A is complete is needed since we will use Matlis-Duality.

Theorem 4.3. *[with hypotheses as in 4.1] For $i = 0, \ldots, s-1$ we have*
(1) *$\ell(H^i(E)_n) < \infty$ for all $n \in \mathbb{Z}$.*
(2) *$H^i(E)^\vee$ is a Noetherian $\mathcal{R}(I)$-module.*
(3) *$H^i(E)$ is a *-Artinian $\mathcal{R}(I)$-module.*

Proof. We prove everything together by induction on s.

The case $s = 1$
Clearly $\ell(H^0(E)_n) < \infty$ for all $n \in \mathbb{Z}$ and is zero for $n \ll 0$. By hypothesis E is quasi-finite of order at least 1. So $H^0(E)_n = 0$ for all $n \gg 0$. The result follows.

We assume the result for $s = r$ and prove for $s = r+1$. Since E is quasi-finite module of order at least $r+1$ it is also quasi-finite module of order at least r. So by induction hypothesis applied to E we have that for $i = 0, \ldots, r-1$ the modules $H^i(E)$ satisfy properties (1), (2) and (3). It remains to prove that $H^r(E)$ satisfies properties (1), (2) and (3).
Let u be E-filter regular. Set $F = E/uE$. We have an exact sequence

$$0 \longrightarrow (0 \colon {}_E u) \longrightarrow E(-1) \xrightarrow{u} E \longrightarrow F \longrightarrow 0.$$

Since $(0 \colon {}_E u)$ is R_+-torsion, by using a standard trick, we get the exact sequence

$$\begin{aligned} 0 \longrightarrow (0 \colon {}_E u) \longrightarrow H^0(E)(-1) &\xrightarrow{u} H^0(E) \longrightarrow H^0(F) \longrightarrow \\ H^1(E)(-1) &\xrightarrow{u} H^1(E) \longrightarrow H^1(F) \longrightarrow \\ &\cdots \\ H^{r-1}(E)(-1) &\xrightarrow{u} H^{r-1}(E) \longrightarrow H^{r-1}(F) \longrightarrow \\ H^r(E)(-1) &\xrightarrow{u} H^r(E). \end{aligned}$$

So we have an exact sequence

$$(*) \qquad H^{r-1}(F) \xrightarrow{\delta} H^r(E)(-1) \xrightarrow{u} H^r(E).$$

Since F is quasi-finite of order at least r we get that $H^{r-1}(F)$ satisfies properties (1), (2) and (3). We prove that $H^r(E)$ satisfies properties (1), (2) and (3).

(1) By hypothesis on E we have $H^r(E)_n = 0$ for all $n \gg 0$ say from $n \geq c+1$. By equation (*) we have $H^{r-1}(F)_{c+1} \xrightarrow{\delta} H^r(E)_c \longrightarrow H^r(E)_{c+1} = 0$. Since $H^{r-1}(F)$ satisfies (1) we get that $H^r(E)_c$ has finite length. Once can induct on j to show that $H^r(E)_{c-j}$ has finite length for all $j \geq 0$.

(2) We have an exact sequence of $\mathcal{R}(I)$-modules

$$H^r(E)^\vee \xrightarrow{u} H^r(E)^\vee(+1) \xrightarrow{\delta^\vee} H^{r-1}(F)^\vee.$$

Set $W = H^r(E)^\vee$. Since $H^{r-1}(F)^\vee$ is finitely generated $\mathcal{R}(I)$-module it follows that $W/uW(+1)$ (and so W/uW) is finitely generated.

Say $V =< \xi_1, \ldots, \xi_m >$ is a $\mathcal{R}(I)$-submodule of W such that $W = V + uW$. We prove $W = V$. This we do degree-wise. By hypothesis on E we have $H^r(E)_n = 0$ for all $n \gg 0$. So $W_n = 0$ for all $n \ll 0$ say from $n < c$. Since $\deg u = 1$ we have $W_c = V_c$. Notice

$$W_{c+1} = V_{c+1} + uW_c = V_{c+1} + uV_c = V_{c+1}.$$

By induction on j it is easy to show $W_{c+j} = V_{c+j}$ for all $j \geq 0$.

(3) This follows from Matlis duality. $\square$

Corollary 4.4. *[with hypotheses as in 4.1] For $i = 0, \ldots, s-1$ set $\mathfrak{a}(E)_i = \operatorname{ann}_{\mathcal{R}(I)} H^i(E)$ and $\mathfrak{q}_i(E) = \mathfrak{a}(E)_i \cap A$. If $H^i(E) \neq 0$ then $\mathfrak{q}_i(E)$ is $\mathfrak{m}$-primary.*

Proof. Fix i with $0 \leq i \leq s-1$. Set $D_i = H^i(E)$ and assume it is non-zero. It is easily checked using Matlis duality that $\operatorname{ann}_{\mathcal{R}(I)} D_i = \operatorname{ann}_{\mathcal{R}(I)} D_i^\vee$.

Notice $D_i^\vee$ is a finitely generated $\mathcal{R}(I)$-module such that $\ell((D_i^\vee)_n)$ is finite for all n. Let $m_1, \ldots, m_s$ be homogeneous generators of $D_i^\vee$. Consider the map

$$\frac{\mathcal{R}(I)}{\mathfrak{a}_i(E)} \xrightarrow{\psi} \bigoplus_{j=1}^{s} D_i^\vee(-\deg m_j)$$
$$\overline{t} \mapsto (tm_1, \ldots, tm_s).$$

Clearly ψ is injective. Taking degree zero part of this embedding gets us that $\mathfrak{q}_i(E)$ is $\mathfrak{m}$-primary. $\square$

5. Proof of main theorem

The proof of the following result is inspired by Theorem 8.1.2 from [3]; (also see [12, Theorem 1]). However we have to be extra careful at a few places. The hypothesis of our result is not exactly similar and we are dealing with infinitely generated modules.

Theorem 5.1. *Let $(A, \mathfrak{m})$ be a complete Noetherian ring with an infinite residue field and let I be an $\mathfrak{m}$-primary ideal in A. Let N be a quasi-finite $\mathcal{R}(I)$-module of order at least m. Assume $\mathbf{u} = u_1, \ldots, u_m \in \mathcal{R}(I)_1$ is a N filter-regular sequence such that*

$$H^*_{\mathbf{u}}(N) = H^*_{\mathcal{R}(I)_+}(N)$$

Also assume that $\ell(N_n)$ is finite for all $n \in \mathbb{Z}$. Set $\mathbf{u}' = u_1, \dots, u_n$ with $n \le m$ and let

$$\mathbb{K}_\bullet = \mathbb{K}_\bullet(\mathbf{u}', N) \colon 0 \to E_n \to \cdots \to E_1 \to E_0 \to 0$$

be the Koszul complex of $\mathbf{u}'$ with coefficients in N.

For $j = 0, \dots, m-1$ set $\mathfrak{b}_j = \operatorname{ann}_{\mathcal{R}(I)} H^j_{\mathcal{R}(I)_+}(N)$ and $\mathfrak{q}_j = A \cap \mathfrak{b}_j$. Then $\mathfrak{q}_0\mathfrak{q}_1 \cdots \mathfrak{q}_{n-1}$ annihilates $H_1(\mathbb{K}_\bullet(\mathbf{u}', N))$.

Proof. Let $\mathbf{C}^\bullet$ be the Cech co-chain complex on $u_1, \dots, u_m$. We shift $\mathbf{C}^\bullet$ m-places and write it as a chain complex

$$\mathbf{D}_\bullet \colon 0 \to D_m \to \cdots \to D_1 \to D_0 \to 0.$$

By construction $H_i(N \otimes \mathbf{D}_\bullet) = H^{m-i}_{\mathcal{R}(I)_+}(N)$.

Consider the chain bicomplex $\mathbf{X} = \mathbf{D}_\bullet \otimes \mathbb{K}_\bullet$. We consider the two standard spectral sequences to compute the homology of $\mathbf{Y}_\bullet = \operatorname{Tot}(\mathbf{X})$; the total complex of $\mathbf{X}$.

The first spectral sequence:
${}^{I}E^0_{pq} = D_p \otimes K_q$. So

$$\begin{aligned} {}^{I}E^1_{pq} &= H_q(D_p \otimes \mathbb{K}_\bullet) \\ &= D_p \otimes H_q(\mathbb{K}_\bullet), \quad \text{since } D_p \text{ is flat.} \end{aligned}$$

By Theorem 3.1 we have that $H_q(\mathbb{K}_\bullet)$ is $\mathcal{R}(I)_+$-torsion for all $q > 0$. It follows that

$${}^{I}E^1_{pq} = \begin{cases} 0 & \text{for } q > 0 \text{ and } p \neq m, \\ H_q(\mathbb{K}_\bullet) & \text{for } q > 0 \text{ and } p = m, \\ D_p \otimes H_0(\mathbb{K}_\bullet) & \text{for } q = 0. \end{cases}$$

Therefore

$${}^{I}E^2_{pq} = \begin{cases} 0 & \text{for } q > 0 \text{ and } p \neq m, \\ H_q(\mathbb{K}_\bullet) & \text{for } q > 0 \text{ and } p = m, \\ H^{m-p}_{\mathcal{R}(I)_+}(H_0(\mathbb{K}_\bullet)) & \text{for } q = 0. \end{cases}$$

Observe that this spectral sequence *collapses* at ${}^{I}E^2$. So $H_{m+i}(\mathbf{Y}_\bullet) \cong H_i(\mathbb{K}_\bullet)$ for $1 \le i \le n$.

The second spectral sequence:
${}^{II}E^0_{pq} = D_q \otimes K_p$. So

$${}^{II}E^1_{pq} = H_q(\mathbf{D}_\bullet \otimes K_p) = H^{m-q}_{R_+}(K_p) = \left(H^{m-q}_{R_+}(N)\right)^{\binom{n}{p}}.$$

By construction $\mathfrak{q}_{m-q}$ annihilates ${}^{II}E^1_{pq}$ if $q \neq 0$. Since ${}^{II}E^\infty_{pq}$ is a subquotient of ${}^{II}E^1_{pq}$ we get that $\mathfrak{q}_{m-q}$ annihilates E^∞_{pq} if $q \neq 0$.

Let $0 = V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{j-1} \subseteq V_j = H_{m+1}(\mathbf{Y}_\bullet)$ be the filtration such that ${}^{II}E^\infty_{p,m+1-p} \cong V_p/V_{p-1}$. Notice ${}^{II}E^\infty_{p,m+1-p} = 0$ for $p > n$ and $m+1-p > m$ (equivalently $p < 1$). So in the filtration $1 \le p \le n$. Notice in this range $q = m+1-p \neq 0$ (otherwise $p = m+1 > n$). So $\mathfrak{q}_{m-q} = \mathfrak{q}_{p-1}$ annihilates ${}^{II}E^\infty_{p,m+1-p}$ for the range $1 \le p \le n$. It follows that $\mathfrak{q}_0\mathfrak{q}_1 \cdots \mathfrak{q}_{n-1}$ annihilates $H_{m+1}(\mathbf{Y}_\bullet)$. The result follows since $H_{m+1}(\mathbf{Y}_\bullet) = H_1(\mathbb{K}_\bullet)$. $\square$

Theorem 5.2. *Let $(A, \mathfrak{m})$ be a complete Cohen-Macaulay local ring of dimension with infinite residue field and dimension $d \ge 1$. Let I be an $\mathfrak{m}$-primary ideal in A.*

Set $L = L^I(A)$. For $i = 0, \ldots, d-1$ set $\mathfrak{q}_i = A \cap \operatorname{ann}_{\mathcal{R}(I)} H^i_{R_+}(L)$. For $r = 1, \ldots, d$ set

$$\mathfrak{a}_r(I) = \bigcap_{\substack{\mathbf{x} = x_1, \ldots, x_r \text{ is a} \\ \text{superficial sequence of } I}} \operatorname{ann}_A \mathcal{V}_I(\mathbf{x})$$

Then $\mathfrak{a}_r(I) \supseteq \mathfrak{q}_0 \cdots \mathfrak{q}_{r-1}$. In particular if $\operatorname{depth} G_I(A) < r$ then $\mathfrak{a}_r(I)$ is $\mathfrak{m}$-primary.

Proof. By 2.4, L is quasi-finite $\mathcal{R}(I)$-module of order at least d. Fix $r \geq 1$. Let $\mathbf{x}' = x_1, \ldots, x_r$ be an I-superficial sequence. Then $\mathbf{x}'$ can be extended to a maximal superficial sequence $\mathbf{x} = x_1, \ldots, x_r, x_{r+1}, \ldots, x_d$. Set $u_i = x_i t \in \mathcal{R}(I)_1$. Then by 2.5 $\mathbf{u} = u_1, \ldots, u_d$ is a L-filter regular sequence. Since $(\mathbf{x})$ is a reduction of I it follows that $\mathbf{u}$ generates $\mathcal{R}(I)_+$ up to radical. So $H^i_{\mathbf{u}}(L) = H^i_{\mathcal{R}(I)_+}(L)$. Set $\mathbf{u}' = u_1, \ldots, u_r$. Let $\mathbb{K}_\bullet(\mathbf{u}', L)$ be the Koszul complex on $\mathbf{u}'$ with coefficients in L. By 3.2 we get that $H_1(\mathbf{u}', L) = \mathcal{V}_I(\mathbf{x}')$. From Theorem 5.1. we get $\operatorname{ann}_A \mathcal{V}_I(\mathbf{x}') \supseteq \mathfrak{q}_0 \cdots \mathfrak{q}_{r-1}$. Since $\mathbf{x}'$ was an arbitary superficial sequence of length r we get $\mathfrak{a}_r(I) \supseteq \mathfrak{q}_0 \cdots \mathfrak{q}_{r-1}$. $\square$

We now drop the assumption that A is complete.

Theorem 5.3. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field and dimension $d \geq 1$. Let I be an $\mathfrak{m}$-primary ideal and let $1 \leq r \leq d$. Then*

$$\mathfrak{a}_r(I\widehat{A}) \cap A \subseteq \mathfrak{a}_r(I).$$

Furthermore if $\operatorname{depth} G_I(A) < r$ then $\mathfrak{a}_r(I)$ is $\mathfrak{m}$-primary.

Proof. Let $\widehat{A}$ be the completion of A. Let $\mathbf{x} = x_1, \ldots, x_r$ be an I-superficial sequence. Then $\mathbf{x}$ considered as a sequence in $\widehat{A}$ is also a $\widehat{I}$-superficial sequence. Furthermore $\mathcal{V}_{I\widehat{A}}(\mathbf{x}) = \mathcal{V}_I(\mathbf{x})$ since it is of finite length. It follows that $\operatorname{ann}_{\widehat{A}} \mathcal{V}_{I\widehat{A}}(\mathbf{x}) \cap A = \operatorname{ann}_A \mathcal{V}_I(\mathbf{x})$.

Notice

$$\mathfrak{a}_r(I\widehat{A}) \subseteq \bigcap_{\substack{\mathbf{x} = x_1, \ldots, x_r \text{ is a} \\ \text{superficial sequence of } I}} \operatorname{ann}_A \mathcal{V}_{I\widehat{A}}(\mathbf{x}).$$

Therefore $\mathfrak{a}_r(I\widehat{A}) \cap A \subseteq \mathfrak{a}_r(I)$. Furthermore as $G_{I\widehat{A}}(\widehat{A}) = G_I(A)$ has depth $< r$ we have that $\mathfrak{a}_r(I\widehat{A})$ is $\widehat{\mathfrak{m}}$-primary. It follows that $\mathfrak{a}_r(I)$ is $\mathfrak{m}$-primary. $\square$

6. Powers of I

In this section we invesitigate $\mathfrak{a}_r(I^l)$ for $l \geq 1$. One of the advantages of $L^I(A)$ is that $L^I(A)(-1)$ commutes with the Veronese functor. Clearly

$$\left(L^I(A)\right)^{<l>} = L^{I^l}(A)(-1) \quad \text{for each } l \geq 1.$$

Also note that for the Rees algebras we have

$$\mathcal{R}(I^l) = \mathcal{R}(I)^{<l>} \quad \text{and} \quad \mathcal{R}(I^l)_+ = \mathcal{R}(I)_+^{<l>}.$$

Local cohomology also commutes with the Veronese functor. So we have that

$$H^i_{\mathcal{R}(I^l)_+}\left(L^{I^l}(A)(-1)\right) \cong \left(H^i_{\mathcal{R}(I)_+}(L^I(A))(-1)\right)^{<l>} \quad \text{for all } l \geq 1.$$

We first prove the following general result.

Lemma 6.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and let I be an $\mathfrak{m}$-primary ideal. Let E be a finitely generated graded $\mathcal{R}(I)$-module with $\ell(E_n) < \infty$ for all $n \in \mathbb{Z}$. For $l \geq 1$ set*

$$\mathfrak{q}(I^l)_E = \left(\operatorname{ann}_{\mathcal{R}(I^l)} E^{<l>}\right) \cap A.$$

Then

(1) *$\mathfrak{q}(I^l)_E$ is $\mathfrak{m}$-primary for each $l \geq 1$.*

(2) *For each $r, l \geq 1$ we have*

$$\mathfrak{q}(I^l)_E \subseteq \mathfrak{q}(I^{rl})_E.$$

(3) *The set*

$$\mathcal{C} = \{\mathfrak{q}(I^l)_E \mid l \geq 1\},$$

has a unique maximal element which we denote as $\mathfrak{q}(I)^\infty_E$.

Proof. (1). Fix $l \geq 1$. Then $E^{<l>}$ is a finitely generated graded $\mathcal{R}(I^l)$-module with $\ell(E^{<l>}_j)$ finite for all $j \in \mathbb{Z}$. So by an argument similar to Corollary 4.4 we have that $\mathfrak{q}(I^l)_E$ is $\mathfrak{m}$-primary.

(2). Notice

$$\left(E^{<l>}\right)^{<r>} = E^{<rl>}.$$

Thus it suffices to prove the result for $l = 1$. Let $a \in \mathfrak{q}(I)_E$. Then $aE_j = 0$ for all $j \in \mathbb{Z}$. So we have that $a \in \operatorname{ann}_{\mathcal{R}(I^r)} E^{<r>}$. Also as $a \in A$ we have that $a \in \mathfrak{q}(I^r)_E$.

(3) Suppose $\mathfrak{q}(I^l)_E$ and $\mathfrak{q}(I^r)_E$ are maximal elements in $\mathcal{C}$. By (2) we have that

$$\mathfrak{q}(I^l)_E \subseteq \mathfrak{q}(I^{rl})_E \quad \text{and} \quad \mathfrak{q}(I^r)_E \subseteq \mathfrak{q}(I^{rl})_E.$$

By maximality of $\mathfrak{q}(I^l)_E$ in $\mathcal{C}$ we have that $\mathfrak{q}(I^l)_E = \mathfrak{q}(I^{rl})_E$. Similarly $\mathfrak{q}(I^r)_E = \mathfrak{q}(I^{rl})_E$. So $\mathfrak{q}(I^l)_E = \mathfrak{q}(I^r)_E$. $\square$

Question 6.2. *(with hypotheses as above) Is*

$$\mathfrak{q}(I)^\infty_E = \mathfrak{q}(I^l)_E \quad \textit{for all } l \gg 0?$$

We now prove the following result:

Theorem 6.3. *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay local ring with infinite residue field and dimension $d \geq 1$. Let I be an $\mathfrak{m}$-primary ideal and let $1 \leq r \leq d$. If $\operatorname{depth} G_I(A) < r$ then*

$$\bigcap_{n \geq 1} \mathfrak{a}_r(I^n) \quad \textit{is } \mathfrak{m}\textit{-primary.}$$

Proof. By Theorem 5.3

$$\mathfrak{a}_r(I\widehat{A}) \cap A \subseteq \mathfrak{a}_r(I).$$

Thus $\mathfrak{a}_r(I^n\widehat{A}) \cap A \subseteq \mathfrak{a}_r(I^n)$ for all $n \geq 1$. Thus it suffices to prove the result when A is complete. Let $l \geq 1$. For $i = 0, 1, \ldots, r-1$, define

$$\mathfrak{q}_i(I^l) = \left(\operatorname{ann}_{\mathcal{R}(I^l)} H^i_{\mathcal{R}(I)_+}(L^{I^l}(A))\right) \cap A.$$

By Theorem 5.2

$$\mathfrak{a}_r(I^l) \supseteq \mathfrak{q}_0(I^l)\mathfrak{q}_1(I^l) \cdots \mathfrak{q}_{r-1}(I^l).$$

For $i = 0, 1, \ldots, r-1$ set

$$D_i(l) = H^i_{\mathcal{R}(I)_+}\left(L^{I^l}(A)(-1)\right)^\vee.$$

Note that by Matlis duality

$$D^i(l)^\vee = H^i_{\mathcal{R}(I)_+}\left(L^{I^l}(A)(-1)\right).$$

Clearly

$$\mathfrak{q}_i(I^l) = \left(\operatorname{ann}_{\mathcal{R}(I^l)} D_i(l)\right) \cap A \quad \text{for } i = 0, 1, \ldots, r-1.$$

Since $L^I(A)$ and local cohomology behaves well with respect to the Veronese functor we have that for all $l \geq 1$ we have

$$D_i(l) = D_i(1)^{<l>} \quad \text{for } i = 0, 1, \ldots, r-1.$$

By Lemma 6.1(2) we have $\mathfrak{q}_i(I^l) \supseteq \mathfrak{q}_i(I)$ for all $l \geq 1$ and for all $i = 0, \ldots, r-1$. Therefore we have

$$\mathfrak{a}_r(I^l) \supseteq \mathfrak{q}_0(I)\mathfrak{q}_1(I)\cdots\mathfrak{q}_{r-1}(I) \text{ for all } l \geq 1.$$

It follows that $\bigcap_{n\geq 1} \mathfrak{a}_r(I^n)$ is $\mathfrak{m}$-primary. ∎

We end our paper with the following:

Question 6.4. *(with hypothesis as above) Is $\mathfrak{a}_r(I^n)$ constant for all $n \gg 0$?*

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